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# Engineering Vibration

FOURTH EDITION

Daniel J. Inman

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# Engineering Vibration

Fourth Edition

DANIEL J. INMAN

*University of Michigan*

International Editions contributions by

RAMESH CHANDRA SINGH

*Delhi Technological University*

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# Contents

# Preface

This book is intended for use in a first course in vibrations or structural dynamics for undergraduates in mechanical, civil, and aerospace engineering or engineering mechanics. The text contains the topics normally found in such courses in accredited engineering departments as set out initially by Den Hartog and refined by Thompson. In addition, topics on design, measurement, and computation are addressed.

## Pedagogy

Originally, a major difference between the pedagogy of this text and competing texts is the use of high level computing codes. Since then, the other authors of vibrations texts have started to embrace use of these codes. While the book is written so that the codes do not have to be used, I strongly encourage their use. These codes (Mathcad®, MATLAB®, and Mathematica®) are very easy to use, at the level of a programmable calculator, and hence do not require any prerequisite courses or training. Of course, it is easier if the students have used one or the other of the codes before, but it is not necessary. In fact, the MATLAB® codes can be copied directly and will run as listed. The use of these codes greatly enhances the student's understanding of the fundamentals of vibration. Just as a picture is worth a thousand words, a numerical simulation or plot can enable a completely dynamic understanding of vibration phenomena. Computer calculations and simulations are presented at the end of each of the first four chapters. After that, many of the problems assume that codes are second nature in solving vibration problems.

Another unique feature of this text is the use of "windows," which are distributed throughout the book and provide reminders of essential information pertinent to the text material at hand. The windows are placed in the text at points where such prior information is required. The windows are also used to summarize essential information. The book attempts to make strong connections to previous course work in a typical engineering curriculum. In particular, reference is made to calculus, differential equations, statics, dynamics, and strength of materials course work.

## WHAT'S NEW IN THIS EDITION

Most of the changes made in this edition are the result of comments sent to me by students and faculty who have used the 3rd edition. These changes consist of improved clarity in explanations, the addition of some new examples that clarify concepts, and enhanced problem statements. In addition, some text material deemed outdated and not useful has been removed. The computer codes have also been updated. However, software companies update their codes much faster than the publishers can update their texts, so users should consult the web for updates in syntax, commands, etc. One consistent request from students has been not to reference data appearing previously in other examples or problems. This has been addressed by providing all of the relevant data in the problem statements. Three undergraduate engineering students (one in Engineering Mechanics, one in Biological Systems Engineering, and one in Mechanical Engineering) who had the prerequisite courses, but had not yet had courses in vibrations, read the manuscript for clarity. Their suggestions prompted us to make the following changes in order to improve readability from the student's perspective:

- Improved clarity in explanations added in 47 different passages in the text. In addition, two new windows have been added.
- Twelve new examples that clarify concepts and enhanced problem statements have been added, and ten examples have been modified to improve clarity.
- Text material deemed outdated and not useful has been removed. Two sections have been dropped and two sections have been completely rewritten.
- All computer codes have been updated to agree with the latest syntax changes made in MATLAB, Mathematica, and Mathcad.
- Fifty-four new problems have been added and 94 problems have been modified for clarity and numerical changes.
- Eight new figures have been added and three previous figures have been modified.
- Four new equations have been added.

*Chapter 1*: Changes include new examples, equations, and problems. New textual explanations have been added and/or modified to improve clarity based on student suggestions. Modifications have been made to problems to make the problem statement clear by not referring to data from previous problems or examples. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

*Chapter 2*: New examples and figures have been added, while previous examples and figures have been modified for clarity. New textual explanations have also been added and/or modified. New problems have been added and older problems modified to make the problem statement clear by not referring to data from previous problems or examples. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

*Chapter 3*: New examples and equations have been added, as well as new problems. In particular, the explanation of impulse has been expanded. In addition, previous problems have been rewritten for clarity and precision. All examples and problems that referred to prior information in the text have been modified to present a more self-contained statement. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

*Chapter 4*: Along with the addition of an entirely new example, many of the examples have been changed and modified for clarity and to include improved information. A new window has been added to clarify matrix information. A figure has been removed and a new figure added. New problems have been added and older problems have been modified with the goal of making all problems and examples more self-contained. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced. Several new plots intermixed in the codes have been redone to reflect issues with Mathematica and MATLAB's automated time step which proves to be inaccurate when using singularity functions. Several explanations have been modified according to students' suggestions.

*Chapter 5*: Section 5.1 has been changed, the figure replaced, and the example changed for clarity. The problems are largely the same but many have been changed or modified with different details and to make the problems more self-contained. Section 5.8 (Active Vibration Suppression) and Section 5.9 (Practical Isolation Design) have been removed, along with the associated problems, to make room for added material in the earlier chapters without lengthening the book. According to user surveys, these sections are not usually covered.

*Chapter 6*: Section 6.8 has been rewritten for clarity and a window has been added to summarize modal analysis of the forced response. New problems have been added and many older problems restated for clarity. Further details have been added to several examples. A number of small additions have been made to the to the text for clarity.

*Chapters 7 and 8*: These chapters were not changed, except to make minor corrections and additions as suggested by users.

## Units

This book uses SI units. The 1st edition used a mixture of US Customary and SI, but at the insistence of the editor all units were changed to SI. I have stayed with SI in this edition because of the increasing international arena that our engineering graduates compete in. The engineering community is now completely global. For instance, GE Corporate Research has more engineers in its research center in India than it does in the US. Engineering in the US is in danger of becoming the 'garment' workers of the next decade if we do not recognize the global work place. Our engineers need to work in SI to be competitive in this increasingly international work place.

### Instructor Support

This text comes with a bit of support. In particular, MS PowerPoint presentations are available for each chapter along with some instructive movies. The solutions manual is available in both MS Word and PDF format (sorry, instructors only). Sample tests are available. The MS Word solutions manual can be cut and pasted into presentation slides, tests, or other class enhancements. These resources can be found at www.pearsoninternationaleditions.com/inman and will be updated often. Please also email me at daninman@umich.edu with corrections, typos, questions, and suggestions. The book is reprinted often, and at each reprint I have the option to fix typos, so please report any you find to me, as others as well as I will appreciate it.

### Student Support

The best place to get help in studying this material is from your instructor, as there is nothing more educational than a verbal exchange. However, the book was written as much as possible from a student's perspective. Many students critiqued the original manuscript, and many of the changes in text have been the result of suggestions from students trying to learn from the material, so please feel free to email me (daninman@umich.edu) should you have questions about explanations. Also I would appreciate knowing about any corrections or typos and, in particular, if you find an explanation hard to follow. My goal in writing this was to provide a useful resource for students learning vibration for the first time.

## ACKNOWLEDGEMENTS

The cover photo of the unmanned air vehicle is provided courtesy of General Atomics Aeronautical Systems, Inc., all rights reserved. Each chapter starts with two photos of different systems that vibrate to remind the reader that the material in this text has broad application across numerous sectors of human activity. These photographs were taken by friends, students, colleagues, relatives, and some by me. I am greatly appreciative of Robert Hargreaves (guitar), P. Timothy Wade (wind mill, Presidential helicopter), General Atomics (Predator), Roy Trifilio (bridge), Catherine Little (damper), Alex Pankonien (FEM graphic), and Jochen Faber of Liebherr Aerospace (landing gear). Alan Giles of General Atomics gave me an informative tour of their facilities which resulted in the photos of their products.

Many colleagues and students have contributed to the revision of this text through suggestions and questions. In particular, Daniel J. Inman, II; Kaitlyn DeLisi; Kevin Crowely; and Emily Armentrout provided many useful comments from the perspective of students reading the material for the first time. Kaitlyn and Kevin checked all the computer codes by copying them out of the book to

make sure they ran. My former PhD students Ya Wang, Mana Afshari, and Amin Karami checked many of the new problems and examples. Dr. Scott Larwood and the students in his vibrations class at the University of the Pacific sent many suggestions and corrections that helped give the book the perspective of a nonresearch insitution. I have implemented many of their suggestions, and I believe the book's explanations are much clearer due to their input. Other professors using the book, Cetin Cetinkaya of Clarkson University, Mike Anderson of the University of Idaho, Joe Slater of Wright State University, Ronnie Pendersen of Aalborg University Esbjerg, Sondi Adhikari of the Universty of Wales, David Che of Geneva College, Tim Crippen of the University of Texas at Tyler, and Nejat Olgac of the University of Conneticut, have provided discussions via email that have led to improvements in the text, all of which are greatly appreciated. I would like to thank the reviewers: Cetin Cetinkaya, Clarkson University; Dr. Nesrin Sarigul-Klijn, University of California–Davis; and David Che, Geneva College.

Many of my former PhD students who are now academics cotaught this course with me and also offered many suggestions. Alper Erturk (Georgia Tech), Henry Sodano (University of Florida), Pablo Tarazaga (Virginia Tech), Onur Bilgen (Old Dominian University), Mike Seigler (University of Kentucky), and Armaghan Salehian (University of Waterloo) all contributed to clarity in this text for which I am grateful. I have been lucky to have wonderful PhD students to work with. I learned much from them.

I would also like to thank Prof. Joseph Slater of Wright State for reviewing some of the new materials, for writing and managing the associated toolbox, and constantly sending suggestions. Several colleagues from government labs and companies have also written with suggestions which have been very helpful from that perspective of practice.

I have also had the good fortune of being sponsored by numerous companies and federal agencies over the last 32 years to study, design, test, and analyze a large variety of vibrating structures and machines. Without these projects, I would not have been able to write this book nor revise it with the appreciation for the practice of vibration, which I hope permeates the text.

Last, I wish to thank my family for moral support, a sense of purpose, and for putting up with my absence while writing.

*Daniel J. Inman*
*Ann Arbor, Michigan*

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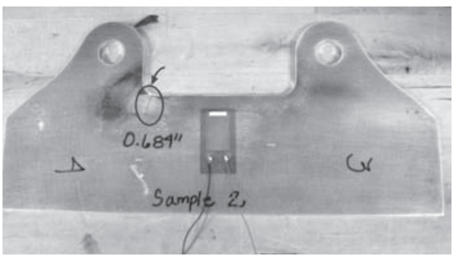
# 1 Introduction to Vibration and the Free Response




Vibration is the subdiscipline of dynamics that deals with repetitive motion. Most of the examples in this text are mechanical or structural elements. However, vibration is prevalent in biological systems and is in fact at the source of communication (the ear vibrates to hear and the tongue and vocal cords vibrate to speak). In the case of music, vibrations, say of a stringed instrument such as a guitar, are desired. On the other hand, in most mechanical systems and structures, vibration is unwanted and even destructive. For example, vibration in an aircraft frame causes fatigue and can eventually lead to failure. An example of fatigue crack is illustrated in the circle in the photo on the bottom left. Everyday experiences are full of vibration and usually ways of mitigating vibration. Automobiles, trains, and even some bicycles have devices to reduce the vibration induced by motion and transmitted to the driver.

The task of this text is to teach the reader how to analyze vibration using principles of dynamics. This requires the use of mathematics. In fact, the sine function provides the fundamental means of analyzing vibration phenomena.

The basic concepts of understanding vibration, analyzing vibration, and predicting the behavior of vibrating systems form the topics of this text. The concepts and formulations presented in the following chapters are intended to provide the skills needed for designing vibrating systems with desired properties that enhance vibration when it is wanted and reduce vibration when it is not.

This first chapter examines vibration in its simplest form in which no external force is present (free vibration). This chapter introduces both the important concept of natural frequency and how to model vibration mathematically.

The Internet is a great source for examples of vibration, and the reader is encouraged to search for movies of vibrating systems and other examples that can be found there.

## 1.1 INTRODUCTION TO FREE VIBRATION

Vibration is the study of the repetitive motion of objects relative to a stationary frame of reference or nominal position (usually equilibrium). Vibration is evident everywhere and in many cases greatly affects the nature of engineering designs. The vibrational properties of engineering devices are often limiting factors in their performance. When harmful, vibration should be avoided, but it can also be extremely useful. In either case, knowledge about vibration—how to analyze, measure, and control it—is beneficial and forms the topic of this book.

Typical examples of vibration familiar to most include the motion of a guitar string, the ride quality of an automobile or motorcycle, the motion of an airplane's wings, and the swaying of a large building due to wind or an earthquake. In the chapters that follow, vibration is modeled mathematically based on fundamental principles, such as Newton's laws, and analyzed using results from calculus and differential equations. Techniques used to measure the vibration of a system are then developed. In addition, information and methods are given that are useful for designing particular systems to have specific vibrational responses.

The physical explanation of the phenomena of vibration concerns the interplay between potential energy and kinetic energy. A vibrating system must have a component that stores potential energy and releases it as kinetic energy in the form of motion (vibration) of a mass. The motion of the mass then gives up kinetic energy to the potential-energy storing device.

Engineering is built on a foundation of previous knowledge and the subject of vibration is no exception. In particular, the topic of vibration builds on previous courses in dynamics, system dynamics, strength of materials, differential equations, and some matrix analysis. In most accredited engineering programs, these courses are prerequisites for a course in vibration. Thus, the material that follows draws information and methods from these courses. Vibration analysis is based on a coalescence of mathematics and physical observation. For example, consider a simple pendulum. You may have seen one in a science museum, in a grandfather clock, or you might make a simple one with a string and a marble. As the pendulum swings back and forth, observe that its motion as a function of time can be described very nicely by the sine function from trigonometry. Even more interesting, if you make a free-body diagram of the pendulum and apply Newtonian mechanics to get the equation of motion (summing moments in this case), the resulting equation of motion has the sine function as its solution. Further, the equation of motion predicts the time it takes for the pendulum to repeat its motion. In this example, dynamics, observation, and mathematics all come into agreement to produce a predictive model of the motion of a pendulum, which is easily verified by experiment (physical observation).

This pendulum example tells the story of this text. We propose a series of steps to build on the modeling skills developed in your first courses in statics, dynamics, and strength of materials combined with system dynamics to find equations of motion of successively more complicated systems. Then we will use the techniques of differential equations and numerical integration to solve these equations of motion to predict how various mechanical systems and structures vibrate. The following example illustrates the importance of recalling the methods learned in the first course in dynamics.

**Example 1.1.1**

Derive the equation of motion of the pendulum in Figure 1.1.

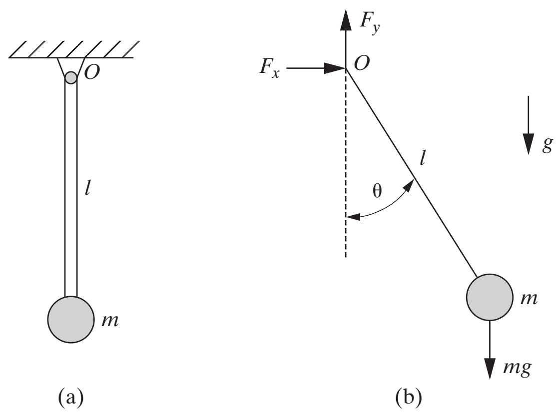



**Figure 1.1** (a) A schematic of a pendulum. (b) The free-body diagram of (a).

**Solution** Consider the schematic of a pendulum in Figure 1.1(a). In this case, the mass of the rod will be ignored as well as any friction in the hinge. Typically, one starts with a photograph or sketch of the part or structure of interest and is immediately faced with having to make assumptions. This is the "art" or experience side of vibration analysis and modeling. The general philosophy is to start with the simplest model possible (hence, here we ignore friction and the mass of the rod and assume the motion remains in a plane) and try to answer the relevant engineering questions. If the simple model doesn't agree with the experiment, then make it more complex by relaxing the assumptions until the model successfully predicts physical observation. With the assumptions in mind, the next step is to create a free-body diagram of the system, as indicated in Figure 1.1(b), in order to identify all of the relevant forces. With all the modeled forces identified, Newton's second law and Euler's second law are used to derive the equations of motion.

In this example Euler's second law takes the form of summing moments about point $O$. This yields

$$\Sigma \mathbf{M}_O = J\boldsymbol{\alpha}$$

where $\mathbf{M}_O$ denotes moments about the point $O$, $J = ml^2$ is the mass moment of inertia of the mass $m$ about the point $O$, $l$ is the length of the massless rod, and $\boldsymbol{\alpha}$ is the angular acceleration vector. Since the problem is really in one dimension, the vector sum of moments equation becomes the single scalar equation

$$J\alpha(t) = -mgl\sin\theta(t) \quad \text{or} \quad ml^2\ddot{\theta}(t) + mgl\sin\theta(t) = 0$$

Here the moment arm for the force $mg$ is the horizontal distance $l\sin\theta$, and the two overdots indicate two differentiations with respect to the time, $t$. This is a second-order ordinary differential equation, which governs the time response of the pendulum. This is exactly the procedure used in the first course in dynamics to obtain equations of motion.

The equation of motion is nonlinear because of the appearance of the $\sin(\theta)$ and hence difficult to solve. The nonlinear term can be made linear by approximating the sine for small values of $\theta(t)$ as $\sin\theta \approx \theta$. Then the equation of motion becomes

$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$

This is a linear, second-order ordinary differential equation with constant coefficients and is commonly solved in the first course of differential equations (usually the third course in the calculus sequence). As we will see later in this chapter, this linear equation of motion and its solution predict the period of oscillation for a simple pendulum quite accurately. The last section of this chapter revisits the nonlinear version of the pendulum equation.

□

Since Newton's second law for a constant mass system is stated in terms of force, which is equated to the mass multiplied by acceleration, an equation of motion with two time derivatives will always result. Such equations require two constants of integration to solve. Euler's second law for constant mass systems also yields two time derivatives. Hence the initial position for $\theta(0)$ and velocity of $\dot{\theta}(0)$ must be specified in order to solve for $\theta(t)$ in Example 1.1.1. The term $mgl\sin\theta$ is called the *restoring force*. In Example 1.1.1, the restoring force is gravity, which provides a potential-energy storing mechanism. However, in most structures and machine parts the restoring force is elastic. This establishes the need for background in strength of materials when studying vibrations of structures and machines.

As mentioned in the example, when modeling a structure or machine it is best to start with the simplest possible model. In this chapter, we model only systems that can be described by a single degree of freedom, that is, systems for which Newtonian mechanics result in a single scalar equation with one displacement coordinate. The degree of freedom of a system is the minimum number of displacement coordinates needed to represent the position of the system's mass at any instant of time. For instance, if the mass of the pendulum in Example 1.1.1 were a rigid body, free to rotate about the end of the pendulum as the pendulum swings, the angle of rotation of the mass would define an additional degree of freedom. The problem would then require two coordinates to determine the position of the mass in space, hence two degrees of freedom. On the other hand, if the rod in Figure 1.1 is flexible, its distributed mass must be considered, effectively resulting in an infinite number of degrees of freedom. Systems with more than one degree of freedom are discussed in Chapter 4, and systems with distributed mass and flexibility are discussed in Chapter 6.

The next important classification of vibration problems after degree of freedom is the nature of the input or stimulus to the system. In this chapter, only the free response of the system is considered. Free response refers to analyzing the vibration of a system resulting from a nonzero initial displacement and/or velocity of the system with no external force or moment applied. In Chapter 2, the response of a single-degree-of-freedom system to a harmonic input (i.e., a sinusoidal applied force) is discussed. Chapter 3 examines the response of a system to a general forcing function (impulse or shock loads, step functions, random inputs, etc.), building on information learned in a course in system dynamics. In the remaining chapters, the models of vibration and methods of analysis become more complex.

The following sections analyze equations similar to the linear version of the pendulum equation given in Example 1.1.1. In addition, energy dissipation is introduced, and details of elastic restoring forces are presented. Introductions to design, measurement, and simulation are also presented. The chapter ends with the introduction of high-level computer codes (MATLAB®, Mathematica, and Mathcad) as a means to visualize the response of a vibrating system and for making the calculations required to solve vibration problems more efficiently. In addition, numerical simulation is introduced in order to solve nonlinear vibration problems.

### 1.1.1 The Spring–Mass Model

From introductory physics and dynamics, the fundamental kinematical quantities used to describe the motion of a particle are displacement, velocity, and acceleration vectors. In addition, the laws of physics state that the motion of a mass with changing velocity is determined by the net force acting on the mass. An easy device to use in thinking about vibration is a spring (such as the one used to pull a storm door shut, or an automobile spring) with one end attached to a fixed object and a mass attached to the other end. A schematic of this arrangement is given in Figure 1.2.

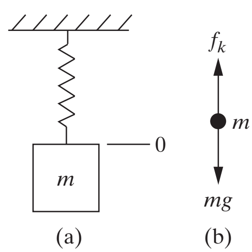



**Figure 1.2** A schematic of (a) a single-degree-of-freedom spring–mass oscillator and (b) its free-body diagram.

Ignoring the mass of the spring itself, the forces acting on the mass consist of the force of gravity pulling down ($mg$) and the elastic-restoring force of the spring pulling back up ($f_k$). Note that in this case the force vectors are collinear, reducing the static equilibrium equation to one dimension easily treated as a scalar. The nature of the spring force can be deduced by performing a simple static experiment. With no mass attached, the spring stretches to the position labeled $x_0 = 0$ in Figure 1.3. As successively more mass is attached to the spring, the force of gravity causes the spring to stretch further. If the value of the mass is recorded, along with the value of the displacement of the end of the spring each time more mass is added, the plot of the force (mass, denoted by $m$, times the acceleration due to gravity, denoted by $g$) versus this displacement, denoted by $x$, yields a curve similar to that illustrated in Figure 1.4. Note that in the region of values for $x$ between 0 and about 20 mm (millimeters), the curve is a straight line. This indicates that for deflections less than 20 mm and forces less than 1000 N (newtons), the force that is applied by the spring to the mass is proportional to the stretch of the spring. The constant of proportionality is the slope of the straight line between 0 and 20 mm. For the particular spring of Figure 1.4, the constant is 50 N/mm, or $5 \times 10^4$ N/m. Thus, the equation that describes the force applied by the spring, denoted by $f_k$, to the mass is the linear relationship

$$f_k = kx \tag{1.1}$$

The value of the slope, denoted by $k$, is called the *stiffness* of the spring and is a property that characterizes the spring for all situations for which the displacement is less than 20 mm. From strength-of-materials considerations, a linear spring of stiffness $k$ stores potential energy of the amount $\frac{1}{2}kx^2$.

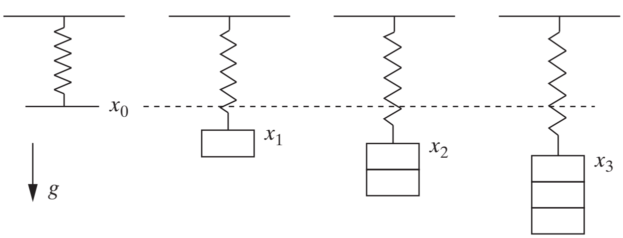



**Figure 1.3** A schematic of a massless spring with no mass attached showing its static equilibrium position, followed by increments of increasing added mass illustrating the corresponding deflections.

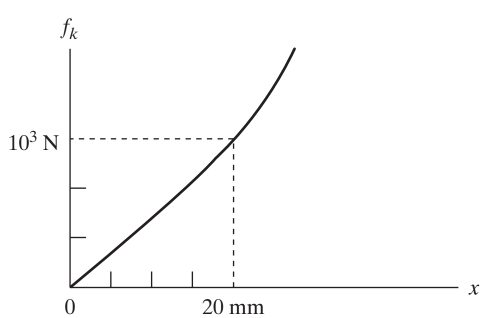



**Figure 1.4** The static deflection curve for the spring of Figure 1.3.

Note that the relationship between $f_k$ and $x$ of equation (1.1) is *linear* (i.e., the curve is linear and $f_k$ depends linearly on $x$). If the displacement of the spring is larger than 20 mm, the relationship between $f_k$ and $x$ becomes *nonlinear*, as indicated in Figure 1.4. Nonlinear systems are much more difficult to analyze and form the topic of Section 1.10. In this and all other chapters, it is assumed that displacements (and forces) are limited to be in the linear range unless specified otherwise.

Next, consider a free-body diagram of the mass in Figure 1.5, with the massless spring elongated from its rest (equilibrium or unstretched) position. As in the earlier figures, the mass of the object is taken to be $m$ and the stiffness of the spring is taken to be $k$. Assuming that the mass moves on a frictionless surface along the $x$ direction, the only force acting on the mass in the $x$ direction is the spring force. As long as the motion of the spring does not exceed its linear range, the sum of the forces in the $x$ direction must equal the product of mass and acceleration.

Summing the forces on the free-body diagram in Figure 1.5 along the $x$ direction yields

$$m\ddot{x}(t) = -kx(t) \qquad \text{or} \qquad m\ddot{x}(t) + kx(t) = 0 \tag{1.2}$$

where $\ddot{x}(t)$ denotes the second time derivative of the displacement (i.e., the acceleration). Note that the direction of the spring force is opposite that of the deflection (+ is marked to the right in the figure). As in Example 1.1.1, the displacement vector and acceleration vector are reduced to scalars, since the net force in the $y$ direction is zero ($N = mg$) and the force in the $x$ direction is collinear with the inertial force. Both the displacement and acceleration are functions of the elapsed time $t$, as denoted in equation (1.2). Window 1.1 illustrates three types of mechanical systems, which for small oscillations can be described by equation (1.2): a spring–mass system, a rotating shaft, and a swinging pendulum (Example 1.1.1). Other examples are given in Section 1.4 and throughout the book.

One of the goals of vibration analysis is to be able to predict the response, or motion, of a vibrating system. Thus it is desirable to calculate the solution to equation (1.2). Fortunately, the differential equation of (1.2) is well known and is covered extensively in introductory calculus and physics texts, as well as in texts on differential equations. In fact, there are a variety of ways to calculate this solution. These are all discussed in some detail in the next section. For now, it is sufficient to present a solution based on physical observation. From experience

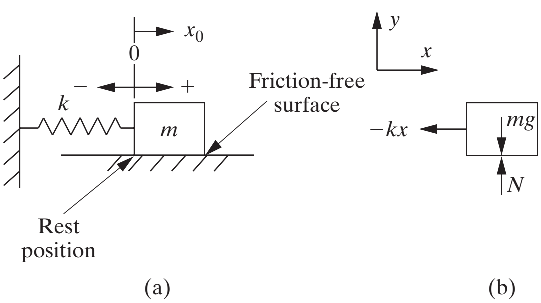
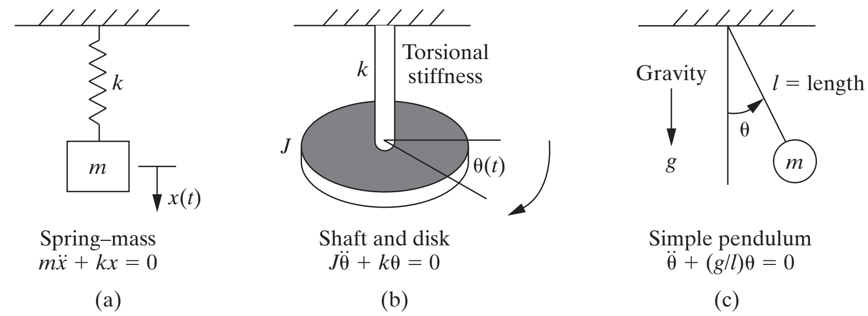



**Figure 1.5** (a) A single spring–mass system given an initial displacement of $x_0$ from its rest, or equilibrium, position and zero initial velocity. (b) The system's free-body diagram.

**Window 1.1**
***Examples of Single-Degree-of-Freedom Systems (for small displacements)***




watching a spring, such as the one in Figure 1.5 (or a pendulum), it is guessed that the motion is periodic, of the form

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.3}$$

This choice is made because the sine function describes oscillation. Equation (1.3) is the sine function in its most general form, where the constant $A$ is the *amplitude*, or maximum value, of the displacement; $\omega_n$, the *angular natural frequency*, determines the interval in time during which the function repeats itself; and $\phi$, called the *phase*, determines the initial value of the sine function. As will be discussed in the following sections, the phase and amplitude are determined by the initial state of the system (see Figure 1.7). It is standard to measure the time $t$ in seconds (s). The phase is measured in radians (rad), and the frequency is measured in radians per second (rad/s). As derived in the following equation, the frequency $\omega_n$ is determined by the physical properties of mass and stiffness ($m$ and $k$), and the constants $A$ and $\phi$ are determined by the initial position and velocity as well as the frequency.

To see if equation (1.3) is in fact a solution of the equation of motion, it is substituted into equation (1.2). Successive differentiation of the displacement, $x(t)$ in the form of equation (1.3), yields the velocity, $\dot{x}(t)$, given by

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi) \tag{1.4}$$

and the acceleration, $\ddot{x}(t)$, given by

$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi) \tag{1.5}$$

Substitution of equations (1.5) and (1.3) into (1.2) yields

$$-m\omega_n^2 A \sin(\omega_n t + \phi) = -kA \sin(\omega_n t + \phi)$$

Dividing by $A$ and $m$ yields the fact that this last equation is satisfied if

$$\omega_n^2 = \frac{k}{m}, \qquad \text{or} \qquad \omega_n = \sqrt{\frac{k}{m}} \tag{1.6}$$

Hence, equation (1.3) is a solution of the equation of motion. The constant $\omega_n$ characterizes the spring–mass system, as well as the frequency at which the motion repeats itself, and hence is called the system's *natural frequency*. A plot of the solution $x(t)$ versus time $t$ is given in Figure 1.6. It remains to interpret the constants $A$ and $\phi$.

The units associated with the notation $\omega_n$ are rad/s and in older texts natural frequency in these units is often referred to as the *circular natural frequency* or *circular frequency* to emphasize that the units are consistent with trigonometric functions and to distinguish this from frequency stated in units of hertz (Hz) or cycles per second, denoted by $f_n$, and commonly used in discussing frequency. The two are related by $f_n = \omega_n/2\pi$ as discussed in Section 1.2. In practice, the phrase *natural frequency* is used to refer to either $f_n$ or $\omega_n$, and the units are stated explicitly to avoid confusion. For example, a common statement is: the natural frequency is 10 Hz, or the natural frequency is $20\pi$ rad/s.

Recall from differential equations that because the equation of motion is of second order, solving equation (1.2) involves integrating twice. Thus there are two constants of integration to evaluate. These are the constants $A$ and $\phi$. The physical significance, or interpretation, of these constants is that they are determined by the initial state of motion of the spring–mass system. Again, recall Newton's laws, if no force is imparted to the mass, it will stay at rest. If, however, the mass is displaced to a position of $x_0$ at time $t = 0$, the force $kx_0$ in the spring will result in motion. Also, if the mass is given an initial velocity of $v_0$ at time $t = 0$, motion will result because

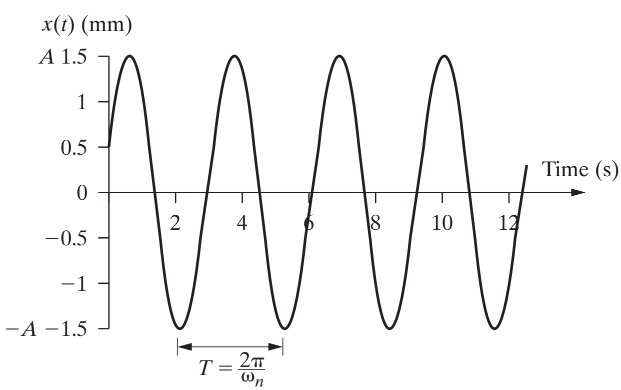



**Figure 1.6** The response of a simple spring–mass system to an initial displacement of $x_0 = 0.5$ mm and an initial velocity of $v_0 = 2\sqrt{2}$ mm/s. The natural frequency is 2 rad/s and the amplitude is 1.5 mm. The period is $T = 2\pi/\omega_n = 2\pi/2 = \pi$s.

of the induced change in momentum. These are called *initial conditions* and when substituted into the solution (1.3) yield

$$x_0 = x(0) = A\sin(\omega_n 0 + \phi) = A\sin\phi \tag{1.7}$$

and

$$v_0 = \dot{x}(0) = \omega_n A\cos(\omega_n 0 + \phi) = \omega_n A\cos\phi \tag{1.8}$$

Solving these two simultaneous equations for the two unknowns $A$ and $\phi$ yields

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \qquad \text{and} \qquad \phi = \tan^{-1}\frac{\omega_n x_0}{v_0} \tag{1.9}$$

as illustrated in Figure 1.7. Here the phase $\phi$ must lie in the proper quadrant, so care must be taken in evaluating the arc tangent. Thus, the solution of the equation of motion for the spring–mass system is given by

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\sin\left(\omega_n t + \tan^{-1}\frac{\omega_n x_0}{v_0}\right) \tag{1.10}$$

and is plotted in Figure 1.6. This solution is called the *free response* of the system, because no force external to the system is applied after $t = 0$. The motion of the spring–mass system is called *simple harmonic motion* or *oscillatory motion* and is discussed in detail in the following section. The spring–mass system is also referred to as a *simple harmonic oscillator*, as well as an *undamped single-degree-of-freedom system*.

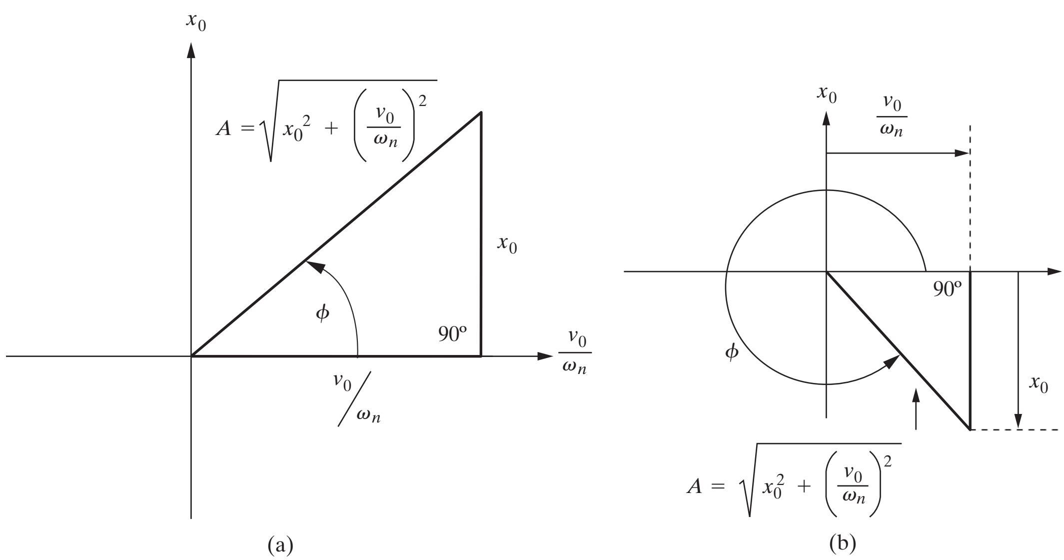



**Figure 1.7** The trigonometric relationships between the phase, natural frequency, and initial conditions. Note that the initial conditions determine the proper quadrant for the phase: (a) for a positive initial position and velocity, (b) for a negative initial position and a positive initial velocity.

**Example 1.1.2**

The phase angle $\phi$ describes the relative shift in the sinusoidal vibration of the spring–mass system resulting from the initial displacement, $x_0$. Verify that equation (1.10) satisfies the initial condition $x(0) = x_0$.

**Solution** Substitution of $t = 0$ in equation (1.10) yields

$$x(0) = A\sin\phi = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\sin\left(\tan^{-1}\frac{\omega_n x_0}{v_0}\right)$$

Figure 1.7 illustrates the phase angle $\phi$ defined by equation (1.9). This right triangle is used to define the sine and tangent of the angle $\phi$. From the geometry of a right triangle, and the definitions of the sine and tangent functions, the value of $x(0)$ is computed to be

$$x(0) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\,\frac{\omega_n x_0}{\sqrt{\omega_n^2 x_0^2 + v_0^2}} = x_0$$

which verifies that the solution given by equation (1.10) is consistent with the initial displacement condition.

□

**Example 1.1.3**

A vehicle wheel, tire, and suspension assembly can be modeled crudely as a single-degree-of-freedom spring–mass system. The (unsprung) mass of the assembly is measured to be about 30 kilograms (kg). Its frequency of oscillation is observed to be 10 Hz. What is the approximate stiffness of the suspension assembly?

**Solution** The relationship between frequency, mass, and stiffness is $\omega_n = \sqrt{k/m}$, so that

$$k = m\omega_n^2 = (30\text{ kg})\left(10\,\frac{\text{cycle}}{\text{s}}\cdot\frac{2\pi\text{ rad}}{\text{cycle}}\right)^2 = 1.184\times 10^5\text{ N/m}$$

This provides one simple way to estimate the stiffness of a complicated device. This stiffness could also be estimated by using a static deflection experiment similar to that suggested by Figures 1.3 and 1.4.

□

**Example 1.1.4**

Compute the amplitude and phase of the response of a system with a mass of 2 kg and a stiffness of 200 N/m, to the following initial conditions:

**a)** $x_0 = 2$ mm and $v_0 = 1$ mm/s
**b)** $x_0 = -2$ mm and $v_0 = 1$ mm/s
**c)** $x_0 = 2$ mm and $v_0 = -1$ mm/s

Compare the results of these calculations.

**Solution** First, compute the natural frequency, as this does not depend on the initial conditions and will be the same in each case. From equation (1.6):

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{N/m}}{2\ \text{kg}}} = 10\ \text{rad/s}$$

Next, compute the amplitude, as it depends on the squares of the initial conditions and will be the same in each case. From equation (1.9):

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{10^2 \cdot 2^2 + 1^2}}{10} = 2.0025\ \text{mm}$$

Thus the difference between the three responses in this example is determined only by the phase. Using equation (1.9) and referring to Figure 1.7 to determine the proper quadrant, the following yields the phase information for each case:

**a)** $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\ \text{rad/s})\ (2\ \text{mm})}{1\ \text{mm/s}}\right) = 1.521\ \text{rad (or } 87.147^\circ)$

which is in the first quadrant.

**b)** $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\ \text{rad/s})\ (-2\ \text{mm})}{1\ \text{mm/s}}\right) = -1.521\ \text{rad (or } -87.147^\circ)$

which is in the fourth quadrant.

**c)** $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\ \text{rad/s})(2\ \text{mm})}{-1\ \text{mm/s}}\right) = (-1.521 + \pi)\ \text{rad (or } 92.85^\circ)$

which is in the second quadrant (position positive, velocity negative places the angle in the second quadrant in Figure 1.7 requiring that the raw calculation be shifted 180°).

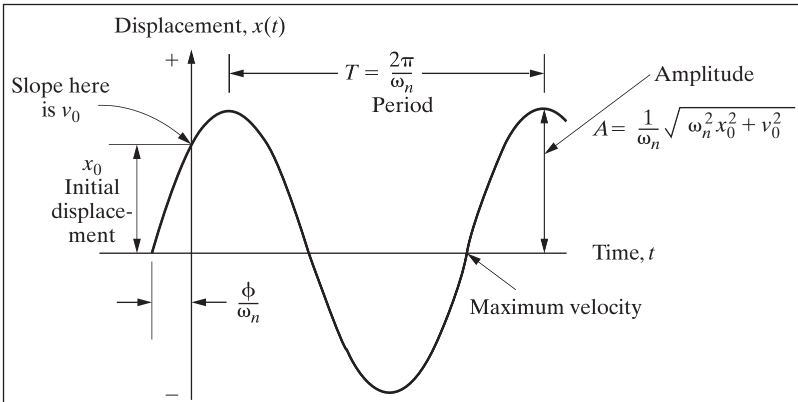
Note that if equation (1.9) is used without regard to Figure 1.7, parts b and c would result in the same answer (which makes no sense physically as the responses each have different starting points). Thus in computing the phase it is important to consider which quadrant the angle should lie in. Fortunately, some calculators and some codes use an arc tangent function, which corrects for the quadrant (for instance, MATLAB uses the `atan2(w0*x0, v0)` command).

The tan($\phi$) can be positive or negative. If the tangent is positive, the phase angle is in the first or third quadrant. If the sign of the initial displacement is positive, the phase angle is in the first quadrant. If the sign is negative or the initial displacement is negative, the phase angle is in the third quadrant. If on the other hand the tangent is negative, the phase angle is in the second or fourth quadrant. As in the previous case, by examining the sign of the initial displacement, the proper quadrant can be determined. That is, if the sign is positive, the phase angle is in the second quadrant, and if the sign is negative, the phase angle is in the fourth quadrant. The remaining possibility is that the tangent is equal to zero. In this case, the phase angle is either zero or 180°. The initial velocity determines which quadrant is correct. If the initial displacement is zero and if the initial velocity is zero, then the phase angle is zero. If on the other hand the initial velocity is negative, the phase angle is 180°.

□

The main point of this section is summarized in Window 1.2. This illustrates harmonic motion and how the initial conditions determine the response of such a system.

**Window 1.2**
***Summary of the Description of Simple Harmonic Motion***



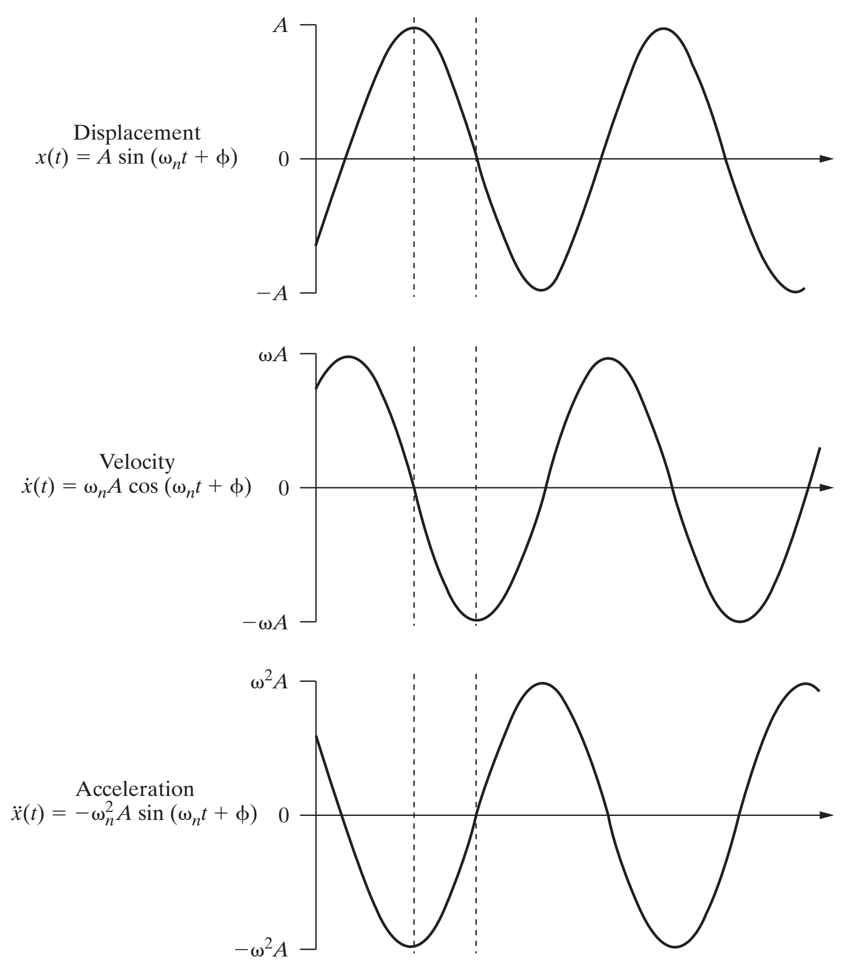

$$x(t) = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}\ \sin(\omega_n t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) \qquad v_0 = \text{initial velocity}$$

## 1.2 HARMONIC MOTION

The fundamental kinematic properties of a particle moving in one dimension are displacement, velocity, and acceleration. For the harmonic motion of a simple spring–mass system, these are given by equations (1.3), (1.4), and (1.5), respectively. These equations reveal the different relative amplitudes of each quantity. For systems with natural frequency larger than 1 rad/s, the relative amplitude of the velocity response is larger than that of the displacement response by a multiple of $\omega_n$, and the acceleration response is larger by a multiple of $\omega_n^2$. For systems with frequency less than 1, the velocity and acceleration have smaller relative amplitudes than the displacement. Also note that the velocity is 90° (or $\pi/2$ radians) out of phase with the position [i.e., $\sin(\omega_n t + \pi/2 + \phi) = \cos(\omega_n t + \phi)$], while the acceleration is 180° out of phase with the position and 90° out of phase with the velocity. This is summarized and illustrated in Window 1.3.

**Window 1.3**
***The Relationship between Displacement, Velocity, and Acceleration for Simple Harmonic Motion***



The angular natural frequency, $\omega_n$, used in equations (1.3) and (1.10), is measured in radians per second and describes the repetitiveness of the oscillation. As indicated in Window 1.2, the time the cycle takes to repeat itself is the *period, T*, which is related to the natural frequency by

$$T = \frac{2\pi \text{ rad}}{\omega_n \text{ rad/s}} = \frac{2\pi}{\omega_n}\text{ s} \tag{1.11}$$

This results from the elementary definition of the period of a sine function. The frequency in hertz (Hz), denoted by $f_n$, is related to the frequency in radians per second, denoted by $\omega_n$:

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi}\text{ (Hz)} \tag{1.12}$$

Equation (1.2) is exactly the same form of differential equation as the linear pendulum equation of Example 1.1.1 and of the shaft and disk of Window 1.1(b). As such, the pendulum will have exactly the same form of solution as equation (1.3), with frequency

$$\omega_n = \sqrt{\frac{g}{l}}\text{ rad/s}$$

The solution of the pendulum equation thus predicts that the period of oscillation of the pendulum is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}\text{ s}$$

where the non-italic s denotes seconds. This analytical value of the period can be checked by measuring the period of oscillation of a pendulum with a simple stopwatch. The period of the disk and shaft system of Window 1.1 will have a frequency and period of

$$\omega_n = \sqrt{\frac{k}{J}}\text{ rad/s} \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{J}{k}}\text{ s}$$

respectively. The concept of frequency of vibration of a mechanical system is the single most important physical concept (and number) in vibration analysis. Measurement of either the period or the frequency allows validation of the analytical model. (If you made a 1-meter pendulum, the period would be about 2 s. This is something you could try at home.)

As long as the only disturbance to these systems is a set of nonzero initial conditions, the system will respond by oscillating with frequency $\omega_n$ and period $T$. For the case of the pendulum, the longer the pendulum, the smaller the frequency and the longer the period. That's why in museum demonstrations of a pendulum, the length is usually very large so that $T$ is large and one can easily see the period (also a pendulum is usually used to illustrate the earth's precession; Google the phrase Foucault Pendulum).

**Example 1.2.1**

Consider a small spring about 30 mm (or 1.18 in) long, welded to a stationary table (ground) so that it is fixed at the point of contact, with a 12-mm (or 0.47-in) bolt welded to the other end, which is free to move. The mass of this system is about $49.2 \times 10^{-3}$ kg (equivalent to about 1.73 ounces). The spring stiffness can be measured using the method suggested in Figure 1.4 and yields a spring constant of $k = 857.8$ N/m. Calculate the natural frequency and period. Also determine the maximum amplitude of the response if the spring is initially deflected 10 mm. Assume that the spring is oriented along the direction of gravity as in Window 1.1. (Ignore the effect of gravity; see below.)

**Solution** From equation (1.6) the natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2 \times 10^{-3} \text{ kg}}} = 132 \text{ rad/s}$$

In hertz, this becomes

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

The period is

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = 0.0476 \text{ s}$$

To determine the maximum value of the displacement response, note from Figure 1.6 that this corresponds to the value of the constant $A$. Assuming that no initial velocity is given to the spring ($v_0 = 0$), equation (1.9) yields

$$x(t)_{\max} = A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = x_0 = 10 \text{ mm}$$

Note that the maximum value of the velocity response is $\omega_n A$ or $\omega_n x_0 = 1320$ mm/s and the acceleration response has maximum value

$$\omega_n^2 A = \omega_n^2 x_0 = 174.24 \times 10^3 \text{ mm/s}^2$$

Since $v_0 = 0$, the phase is $\phi = \tan^{-1}(\omega_n x_0/0) = \pi/2$, or 90°. Hence, in this case, the response is $x(t) = 10 \sin(132t + \pi/2) = 10 \cos(132t)$ mm.

□

Does gravity matter in spring problems? The answer is no, if the system oscillates in the linear region. Consider the spring of Figure 1.3 and let a mass of value $m$ extend the spring. Let $\Delta$ denote the distance deflected in this static experiment ($\Delta$is called the static deflection); then the force acting upon the mass is $k\Delta$. From static equilibrium the forces acting on the mass must be zero so that (taking positive down in the figure)

$$mg - k\Delta = 0$$

Next, sum the forces along the vertical for the mass at some point $x$ and apply Newton's law to get

$$m\ddot{x}(t) = -k(x + \Delta) + mg = -kx + mg - \Delta k$$

Note the sign on the spring term is negative because the spring force opposes the motion, which is taken here as positive down. The last two terms add to zero ($mg - k\Delta = 0$) because of the static equilibrium condition, and the equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0$$

Thus gravity does not affect the dynamic response. Note $x(t)$ is measured from the elongated (or compressed if upside down) position of the spring–mass system, that is, from its rest position. This is discussed again using energy methods in Figure 1.14.

**Example 1.2.2**

(a) A pendulum in Brussels swings with a period of 3 seconds. Compute the length of the pendulum. (b) At another location, assume the length of the pendulum is known to be 2 meters and suppose the period is measured to be 2.839 seconds. What is the acceleration due to gravity at that location?

**Solution** The relationship between period and natural frequency is given in equation (1.11). (a) Substitution of the value of natural frequency for a pendulum and solving for the length of the pendulum yields

$$T = \frac{2\pi}{\omega_n} \Rightarrow \omega_n^2 = \frac{g}{l} = \frac{4\pi^2}{T^2} \Rightarrow l = \frac{gT^2}{4\pi^2} = \frac{(9.811\ \mathrm{m/s^2})(3)^2\mathrm{s}^2}{4\pi^2} = 2.237\,\mathrm{m}$$

Here the value of $g = 9.811\ \mathrm{m/s^2}$ is used, as that is the value it has in Brussels (at 51° latitude and an altitude of 102 m). (b) Next, manipulate the pendulum period equation to solve for $g$. This yields

$$\frac{g}{l} = \frac{4\pi^2}{T^2} \Rightarrow g = \frac{4\pi^2}{T^2}l = \frac{4\pi^2}{(2.839)^2\mathrm{s}^2}(2)\mathrm{m} = 9.796\ \mathrm{m/s^2}$$

This is the value of the acceleration due to gravity in Denver, Colorado, United States (at an altitude 1638 m and latitude 40°).

These sorts of calculations are usually done in high school science classes but are repeated here to underscore the usefulness of the concept of natural frequency and period in terms of providing information about the vibration system's physical properties. In addition, this example serves to remind the reader of a familiar vibration phenomenon.

□

The solution given by equation (1.10) was developed assuming that the response should be harmonic based on physical observation. The form of the response can also be derived by a more analytical approach following the theory of elementary differential equations (see, e.g., Boyce and DiPrima, 2009). This approach is reviewed here and will be generalized in later sections and chapters to solve for the response of more complicated systems.

Assume that the solution $x(t)$ is of the form

$$x(t) = ae^{\lambda t} \tag{1.13}$$

where $a$ and $\lambda$ are nonzero constants to be determined. Upon successive differentiation, equation (1.13) becomes $\dot{x}(t) = \lambda ae^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 ae^{\lambda t}$. Substitution of the assumed exponential form into equation (1.2) yields

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \tag{1.14}$$

Since the term $ae^{\lambda t}$ is never zero, expression (1.14) can be divided by $ae^{\lambda t}$ to yield

$$m\lambda^2 + k = 0 \tag{1.15}$$

Solving this algebraically results in

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \tag{1.16}$$

where $j = \sqrt{-1}$ is the imaginary number and $\omega_n = \sqrt{k/m}$ is the natural frequency as before. Note that there are two values for $\lambda$, $\lambda = +\omega_n j$ and $\lambda = -\omega_n j$, because the equation for $\lambda$ is of second order. This implies that there must be two solutions of equation (1.2) as well. Substitution of equation (1.16) into equation (1.13) yields that the two solutions for $x(t)$ are

$$x(t) = a_1 e^{+j\omega_n t} \qquad \text{and} \qquad x(t) = a_2 e^{-j\omega_n t} \tag{1.17}$$

Since equation (1.2) is linear, the sum of two solutions is also a solution; hence, the response $x(t)$ is of the form

$$x(t) = a_1 e^{+j\omega_n t} + a_2 e^{-j\omega_n t} \tag{1.18}$$

where $a_1$ and $a_2$ are complex-valued constants of integration. The Euler relations for trigonometric functions state that $2j \sin\theta = (e^{\theta j} - e^{-\theta j})$ and $2\cos\theta = (e^{\theta j} + e^{-\theta j})$, where $j = \sqrt{-1}$. [See Appendix A, equations (A.18), (A.19), and (A.20), as well as Window 1.5.] Using the Euler relations, equation (1.18) can be written as

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.19}$$

where $A$ and $\phi$ are real-valued constants of integration. Note that equation (1.19) is in agreement with the physically intuitive solution given by equation (1.3). The relationships among the various constants in equations (1.18) and (1.19) are given in Window 1.4. Window 1.5 illustrates the use of Euler relations for deriving harmonic functions from exponentials for the underdamped case.